3.4: Proof by Induction

In this section we will apply the proof by induction to the more general recursively defined objects in the previous section.

Question 1. Consider the pattern of numbers below. Notice that all the rows are palindromic. Will all the subsequent row be palindromic? Explain why or why not.

$$
\begin{array}{c}
1 \\
1\ 1 \\
1\ 0\ 1 \\
1\ 1\ 1\ 1 \\
1\ 0\ 0\ 0\ 1 \\
1\ 1\ 0\ 0\ 1\ 1 \\
1\ 0\ 1\ 0\ 1\ 0\ 1 \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\
1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\
1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\
1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \\
1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1 \\
1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1
\end{array}
$$

Question 2. For an undirected graph $G = (V, E)$, let $c(G) = |V| - |E|$; i.e. the number of vertices of G minus the number of edges of G. Given a graph G, suppose you form a new graph G' by adding a vertex to G in the middle of one of the edges. Notice that $c(G) = c(G')$. Explain why adding a vertex in this manner will always preserve $c(G)$.

To prove a statement using the principle of mathematical induction, you should think of the object in question as being defined using recursion. For a an object $R(n)$ defined for $n \geq 1$ (with base case $n = 1$), the argument to prove Statement(n) will follow as below:

Base Case. Prove Statement(1).
Inductive Case. Prove Statement($k-1$) $\Rightarrow$ Statement(k) for all $k > 1$.

Theorem 1. *For any* $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Theorem 2. *Let $K(1)$, $K(2)$, ... be the sequence of shapes whose limit is the Koch Snowflake. Prove that $K(n)$ is composed of $3 \cdot 4^{n-1}$ lines segments.*

Example 1. Find a formula for $(a_1 \cdots a_n)^R$ for any string of length $n \geq 1$.

Strong Induction. An equivalent variant of proof by induction is what is "strong induction." It is equivalent, so the adjective "strong" is a bit grandiose. With this type of induction we do the following:

Base Case. Prove Statement(1).
Inductive Case. Prove Statement(1) $\wedge \cdots \wedge$ Statement($k-1$) $\Rightarrow$ Statement(k) for all $k > 1$.

Example 2. Show that every binary tree is connected and has no simple circuits.

No Homework. For practice, go through all recursively defined objects in the previous section and prove something about them inductively. See the exercises of Section 3.4 for hints on what to prove.

Group Work 1. For $n \geq 1$, prove that the n^{th} Fibonacci number is

$$F(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Structural Induction. In some cases it is difficult to specify a quantity to do the induction on. Do not fret! The induction follows almost identically.

Group Work 2. Define a set $X \subseteq \mathbb{Z}$ recursively as

Base Case. $4 \in X$
Recursive Case 1. If $x \in X$, then $x - 12 \in X$.
Recursive Case 2. If $x \in X$ then $x^2 \in X$.

Prove that every element of X is divisible by 4.